Answer Key

NAME:

Math 150 Exam 2

Instructions: WRITE YOUR NAME CLEARLY. Do as many problems as you can for a maximal score of 100. SHOW YOUR WORK!

1. If $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

[10 pts]

$F'(x) = f'(g(x)) \cdot g'(x)$ so

$F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot g'(5) = 4 \cdot 6 = \boxed{24.}$

2. Use chain rule to find the derivative of $y = \left(\frac{x^2+1}{x^2-1}\right)^3$ [10 pts]

$y = \left(\frac{x^2-1+2}{x^2-1}\right)^3 = \left(1 + \frac{2}{x^2-1}\right)^3 = \left(1 + 2(x^2-1)^{-1}\right)^3$

$y' = 3\left(1+2(x^2-1)^{-1}\right)^2 \cdot 2(-1) \cdot 2x\,(x^2-1)^{-2} =$

$= 3\left(\frac{x^2+1}{x^2-1}\right)^2 \cdot \frac{-4x}{(x^2-1)^2} = \boxed{\frac{-12x\,(x^2+1)^2}{(x^2-1)^4}}$

3. Let $y(x)$ be given implicitly by the equation $e^{x/y} = x - y$. Find $\frac{dy}{dx}$

[10 pts]

$$\frac{d}{dx}\left(e^{x/y}\right) = \frac{d}{dx}(x-y)$$

$$e^{x/y} \cdot \frac{y - xy'}{y^2} = 1 - y' \Rightarrow e^{\frac{x}{y}}\left(\frac{1}{y} - \frac{x}{y^2}y'\right) = 1 - y'$$

$$\frac{e^{x/y}}{y} - 1 = \left(\frac{xe^{x/y}}{y^2} - 1\right)y'$$

$$\frac{e^{x/y} - y}{y} = \frac{xe^{x/y} - y^2}{y^2}\,y'$$

$$\frac{e^{x/y} - y}{y} \cdot \frac{y^2}{xe^{x/y} - y^2} = y'$$

$$\boxed{\frac{y\left(e^{x/y} - y\right)}{xe^{x/y} - y^2} = y'}$$

4. Find the derivative for the function $y = x^{\sin x}$. [Hint: Use logarithmic differentiation]

[10 pts]

$$y = x^{\sin x} \Rightarrow \ln y = \ln\left(x^{\sin x}\right) = \sin x \ln x$$

$$\Rightarrow \frac{1}{y} \cdot y' = \cos x \ln x + \frac{\sin x}{x}$$

$$y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right) = \boxed{x^{\sin x}\left(\ln\left(x^{\cos x}\right) + \frac{\sin x}{x}\right)}$$

5. A sample of tritium-3 decayed to 94.5% of its original amount after a year.
   (i) What is the half-life of tritium-3? [6 pts]
   (j) How long would it take the sample to decay to 20% of its original amount? [4 pts]

suppose $f(t) = A_0 e^{kt}$ is the decay function of tritium-3, where A_0 is the original amount. Suppose that we wait T years to see the substance decay to $\frac{1}{2}A_0$. Then, starting from any moment t, if we wait additional T years, we observe that

$$f(t+T) = A_0 e^{k(t+T)} = A_0 e^{kt+kT} = A_0 e^{kT} e^{kt} = (A_0 e^{kT}) e^{kt} =$$
$$= \left(\tfrac{1}{2}A_0\right) e^{kt} = \tfrac{1}{2} f(t).$$
Hence, half-life does not depend on the initial amount of the radioactive material A_0.

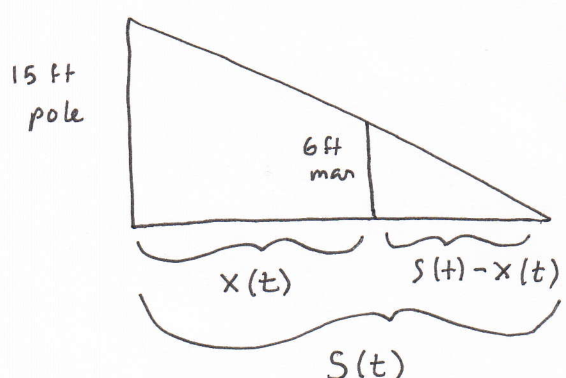
(i) $0.945 = e^{k \cdot 1} \Rightarrow k = \ln(0.945)$

$$\frac{1}{2} = 0.5 = e^{kT} \Rightarrow \ln(0.5) = kT \Rightarrow T = \frac{\ln(0.5)}{\ln(0.945)} \approx 12.25$$

$\boxed{\text{Half-life} \approx 12.25 \text{ years}}$

(ii) let $T_{0.2}$ be the time it takes tritium-3 to decay to 20% the original amount, Then $0.2 = e^{kT_{0.2}} \Rightarrow T_{0.2} = \frac{\ln(0.2)}{\ln(0.945)} \approx \boxed{28.45 \text{ years}}$

6. A street light is mounted at the top of a 15-ft tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole? [10 pts]



$x(t) \equiv$ distance of man from pole at time t.

$s(t) \equiv$ distance of the tip of shadow from pole.

By similarity of triangles $\frac{s(t)}{15} = \frac{s(t) - x(t)}{6}$

Thus $s(t) = \frac{5}{3} x(t)$ and $s'(t) = \frac{5}{3} x'(t) = \frac{5}{3} \cdot 5 = \boxed{\frac{25}{3} \text{ ft/s}}$

7. Use linear approximation to estimate the value of $e^{-0.015}$ [10 pts]

$$e^{a+\Delta x} \approx e^a + e^a(\Delta x) \quad \text{so}$$

$$e^{-0.015} \approx e^0 + e^0(-0.015) = 1 - 0.015 = \boxed{0.985}$$

8. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ for all x > 0 [10 pts]

Let $f(x) = \sqrt{x}$. Then f is everywhere differentiable on $(0, \infty)$. So, by the mean-value-theorem

$$\frac{f(1+x) - f(1)}{x} = \frac{\sqrt{1+x} - \sqrt{1}}{x} = \frac{\sqrt{1+x} - 1}{x} = f'(c)$$

$$= \frac{1}{2\sqrt{c}}$$ for some $1 < c < 1+x$. But then

$$\frac{1}{2\sqrt{c}} < \frac{1}{2\sqrt{1}} = \frac{1}{2}$$ so $$\frac{\sqrt{1+x} - 1}{x} < \frac{1}{2}$$ and hence

$\sqrt{1-x} < 1 + \frac{1}{2}x$ as desired.

9. Calculate $\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$ [10 pts]

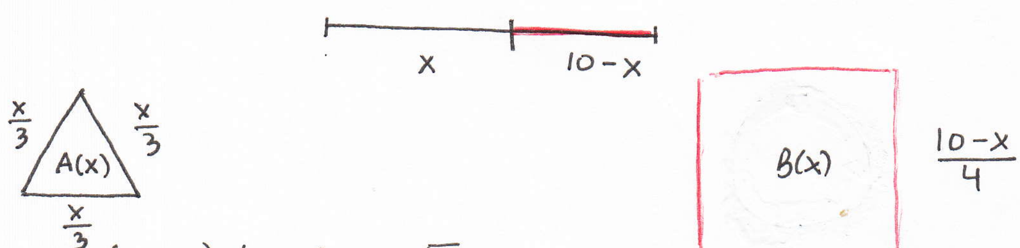
$$\lim_{x\mapsto 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} = \lim_{x\mapsto 0}\frac{\sqrt{1+2x}-1}{x} - \lim_{x\mapsto 0}\frac{\sqrt{1-4x}-1}{x}$$

$$= 2\lim_{x\mapsto 0}\frac{\sqrt{1+2x}-1}{2x} + 4\lim_{x\mapsto 0}\frac{\sqrt{1-4x}-1}{-4x} =$$

$$= \frac{2}{2\sqrt{1}} + \frac{4}{2\sqrt{1}} = \boxed{3}$$

You can also apply L'Hospital's Rule.

10. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is a maximum? How should the wire be cut so that the area is minimal? [10 pts]



$$A(x) = \left(\frac{\sqrt{3}}{6}x\right)\left(\frac{1}{6}x\right) = \frac{\sqrt{3}}{36}x^2 \qquad B(x) = \left(\frac{10-x}{4}\right)^2$$

Set $H(x) = \frac{\sqrt{3}}{36}x^2 + \left(\frac{10-x}{4}\right)^2 = A(x)+B(x).$; $x \in [0,10]$.

Now $H'(x)=0$ if $x = \frac{90}{9+4\sqrt{3}}$ and $\boxed{H(0) \approx 6.25}$ $H(10) \approx 4.81$

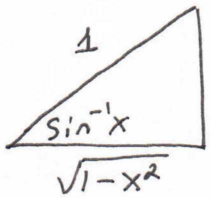
and $\boxed{H\left(\frac{90}{9+4\sqrt{3}}\right) \approx 2.72}$ ← min max.

Extra-Credit

11. Establish the derivative formula for the function $y = \sin^{-1} x$ by using implicit differentiation. [10 pts]

$y = \sin^{-1} x \Rightarrow \sin y = x \Rightarrow y' \cos y = 1 \Rightarrow y' = \frac{1}{\cos y}$

$= \frac{1}{\cos(\sin^{-1} x)}$



$\cos(\sin^{-1} x) = \sqrt{1-x^2}$ Hence $\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

12. Find a function f, whose nth derivative at x = 0 is $f^{(n)}(0) = 5^n n!$. [10 pts]

Represent $f(x)$ as the infinite polynomial $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

we know that $a_n = \frac{f^{(n)}(0)}{n!} = \frac{5^n n!}{n!} = 5^n$. Thus

$f(x) = \sum_{n=0}^{\infty} 5^n x^n = \sum_{n=0}^{\infty} (5x)^n = 1 + (5x) + (5x)^2 + (5x)^3 + \dots + =$

$= 1 + (5x)\left(1 + (5x) + (5x)^2 + \dots +\right) = 1 + (5x) f(x)$. In other words,

$f(x) = 1 + 5x f(x)$ so $f(x) - 5x f(x) = 1$ and $f(x)(1-5x) = 1$

In particular $\boxed{f(x) = \frac{1}{1-5x}}$

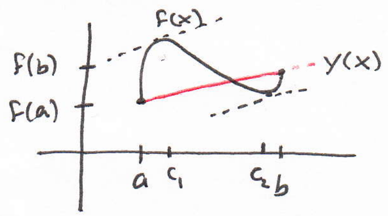
13. State and prove the Mean-Value-Theorem. [10 pts]

Let $f: [a,b] \mapsto \mathbb{R}$ be

1) Cont. on $[a,b]$

2) Diff. on (a,b)

Then there is <u>at least one</u> number $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.



$y(x) \equiv$ secant line through $(a, f(a))$ and $(b, f(b))$

$$y(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Define $H(x) = f(x) - y(x)$. Then $H(a) = H(b) = 0$. Furthermore H is

1) Cont. on $[a,b]$

2) Diff. on (a,b)

Hence, by Rolle's theorem $0 = H'(c) = f'(c) - y'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$ for some $c \in (a,b)$. In particular, $f'(c) = \frac{f(b)-f(a)}{b-a}$.

14. Suppose $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Show that for every integer p, $f(p) = [f(1)]^p$. [10 pts]

1) $f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots +$

$\Rightarrow f'(x) = 1 + \frac{2}{2}x + \frac{3x^3}{3!} + \frac{4x^3}{4!} + \ldots + = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + = f(x)$.

Hence $f'(x) = f(x)$. Also notice that $f(0) = 1 + 0 + \frac{0^2}{2} + \ldots + = 1$.

2) For any real number $a \in \mathbb{R}$ define $g_a(x) = f(a-x)f(x)$. Then $g_a'(x) = -f(a-x)f(x) + f(a-x)f(x) = 0$. $\Rightarrow g_a(x) = C$ for some constant c and all x. In particular, $g_a(0) = f(a-0)f(0) = f(a)$. Hence $f(a-x)f(x) = f(a)$. Setting $a=0$ we have $f(-x)f(x) = f(0) = 1$. Thus $f(-x) = \frac{1}{f(x)}$ for all x.

3) $f(a-[a+b])f(a+b) = f(a) \Rightarrow f(-b)f(a+b) = f(a) \Rightarrow f(a+b) = f(a)f(b)$.

Therefore $f(p) = f(1+[p-1]) = f(1)f(p-1) = f(1)f(1+[p-2]) = [f(1)]^2 f(p-2) = \ldots$
$= [f(1)]^p$.